Deterministic stick-slip dynamics in a one-dimensional random potential

Tim Sullivan & Florian Theil
t.j.sullivan@warwick.ac.uk & f.theil@warwick.ac.uk

Mathematics Institute, University of Warwick

Analysis & numerics for rate-independent processes
Mathematisches Forchungsinstitut Oberwolfach
1 March 2007

Outline of talk

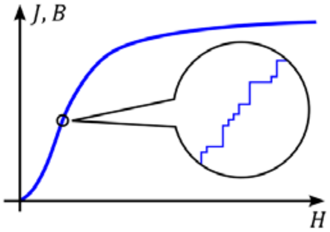
- Introduction and motivation.
- Brief review of (some) previous studies in the area.
- Statement of main result.
- Sketch of the proof of the main result.
- Directions for future research.

Motivation

- Many physical processes exhibit “frictional/stick-slip behaviour”.
- Simple examples:
  - Ball rolling/person skiing down a slope with some bumps.
  - Progression of a dislocation line in a crystal.
  - Evolution of a magnetic domain under an applied field (Barkhausen effect).

Motivation

- Many physical processes exhibit "frictional/stick-slip behaviour".
- Simple examples:
  - Ball rolling/person skiing down a slope with some bumps.
  - Progression of a dislocation line in a crystal.
  - Evolution of a magnetic domain under an applied field (Barkhausen effect).



Motivation

- Intuition suggests that stick-slip behaviour arises from microstructural variations.
- Microstructure $\rightsquigarrow$ macroscopic observables, e.g. yield stresses, coefficients of friction & c.
- These “macro” quantities can be used as parameters in (relatively) successful models, e.g. rate-independent differential inclusions.
- Exactly *how* the microstructure determines macroscopic behaviour is still not generally understood.

Existing approaches

- Rate-independent solutions to differential inclusions:

$$-\nabla V(X_t) + f(t) \in \partial\psi(\dot{X}_t)$$

  with ψ convex and homogeneous of degree one.
- Over-damped limit (neglect kinetic energy):

$$\dot{X}_t^\varepsilon = -\nabla V^\varepsilon (t, X_t^\varepsilon) \,.$$

- Is it possible to extract the first model from the second as a suitable limit as $\varepsilon \downarrow 0$?

Previous one-dimensional studies

$$\dot{X}_t^\varepsilon = -V'(X_t) - (\varepsilon G)'\left(\frac{X_t^\varepsilon}{\varepsilon}\right) + f(\varepsilon t).$$

- Abeyaratne-Chu-James (1996), Menon (2002): averaging methods for periodic perturbations of the potential; not rate-independent, but can extract a rate-independent corollary (limit satisfies a deterministic ordinary differential inclusion determined by bounds on G').
- Grunewald (2005): perturbation is an (integrated) Ornstein-Uhlenbeck process; Fokker-Planck methods insufficient to establish stick-slip behaviour in the limit.

Model in one dimension — random ODE

On the real line $\mathbb{R}$, consider

- a potential $V(x) = \frac{\kappa}{2}x^2$, $\kappa > 0$;
- a C^0 gradient field $g := G' : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$;
  - $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space;
  - wiggly potential $x \mapsto V(x) + \varepsilon G\big(\omega, \frac{x}{\varepsilon}\big)$;
- a C^0 external loading $f : [0, +\infty) \to \mathbb{R}$;

Model in one dimension — random ODE

On the real line $\mathbb{R}$, consider

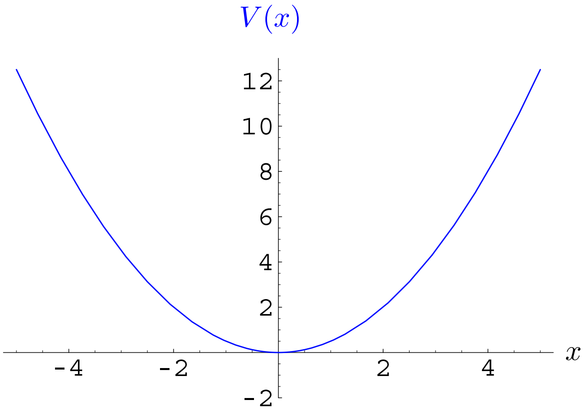
- a potential $V(x) = \frac{\kappa}{2}x^2$, $\kappa > 0$;
- a C^0 gradient field $g := G' : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$;
  - $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space;
  - wiggly potential $x \mapsto V(x) + \varepsilon G\big(\omega, \frac{x}{\varepsilon}\big)$;
- a C^0 external loading $f : [0, +\infty) \to \mathbb{R}$;

Random gradient flow ODE with "landscape parameter" $\omega \in \Omega$:

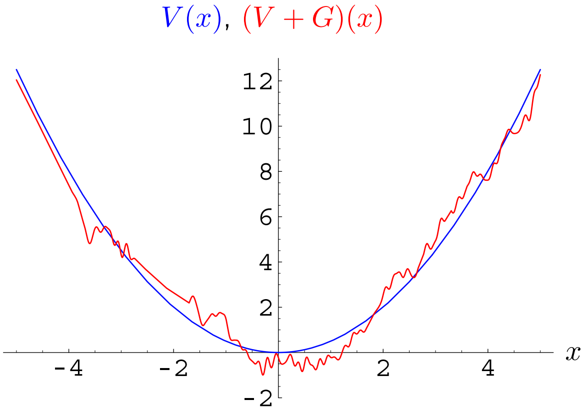
$$\dot{X}_t^\varepsilon(\omega) = -V'\big(X_t^\varepsilon(\omega)\big) - G'\left(\omega, \frac{X_t^\varepsilon(\omega)}{\varepsilon}\right) + f(\varepsilon t).$$

Standard results give existence of solutions for all positive time. Later results remove any need for uniqueness.

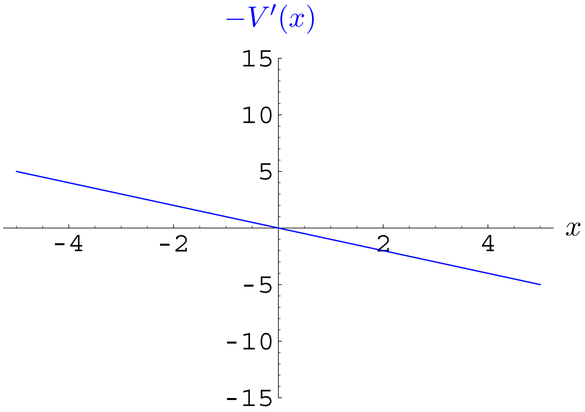
Model in one dimension — random landscape



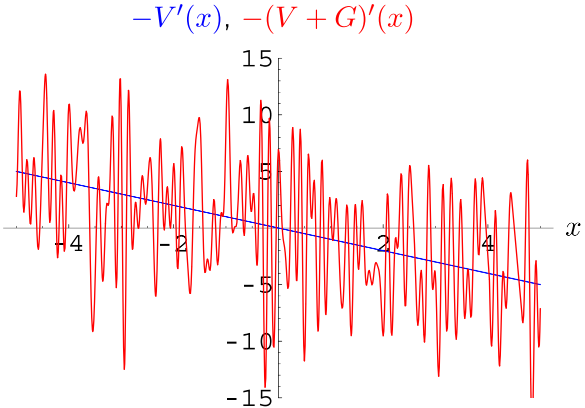
Model in one dimension — random landscape



Model in one dimension — random landscape



Model in one dimension — random landscape



Model in one dimension — limit process

Random ODE:

$$\dot{X}_t^\varepsilon(\omega) = -\kappa X_t^\varepsilon(\omega) - g\left(\omega, \frac{X_t^\varepsilon(\omega)}{\varepsilon}\right) + f(\varepsilon t).$$

Limiting process as $\varepsilon \downarrow 0$:

$$X_t^0 := \lim_{\varepsilon \downarrow 0} X_{t/\varepsilon}^\varepsilon.$$

In principle, this limiting object is a stochastic process $X^0 : \Omega \times [0, +\infty) \to \mathbb{R}$ dependent on the choice of process g and the "landscape parameter" $\omega \in \Omega$, but...

Main theorem — first draft

Theorem (T.J.S.–F.T. (2006))

Let $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ *be a doubly-reflected Wiener process and let* $f \in C^0([0, +\infty); \mathbb{R})$. *Then, for* $\mathbb{P}$*-a.a.* $\omega \in \Omega$, $X^0(\omega)$ *satisfies the deterministic ordinary differential inclusion*

$$-V'(X_t^0) + f(t) \in \partial\psi^\gamma(\dot{X}_t^0), \qquad \text{(ODI)}$$

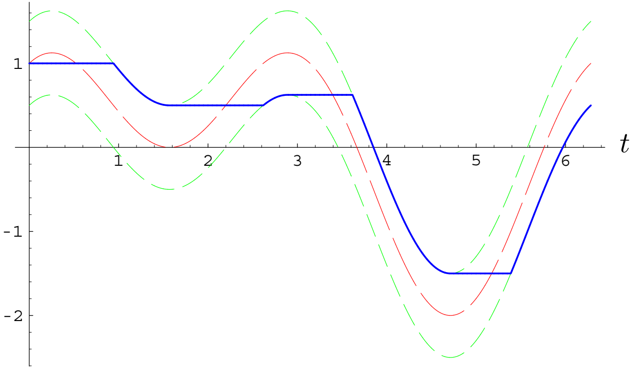
where the dissipation $\psi^\gamma : \mathbb{R} \to [0, +\infty)$ *is given by*

$$\psi^\gamma(\dot{x}) := \begin{cases} \gamma^- \dot{x}; & \dot{x} \leq 0; \\ \gamma^+ \dot{x}; & \dot{x} \geq 0. \end{cases}$$

Note that (ODI) is deterministic and has a unique deterministic solution, which can be easily visualised by the "drainpipe rule".

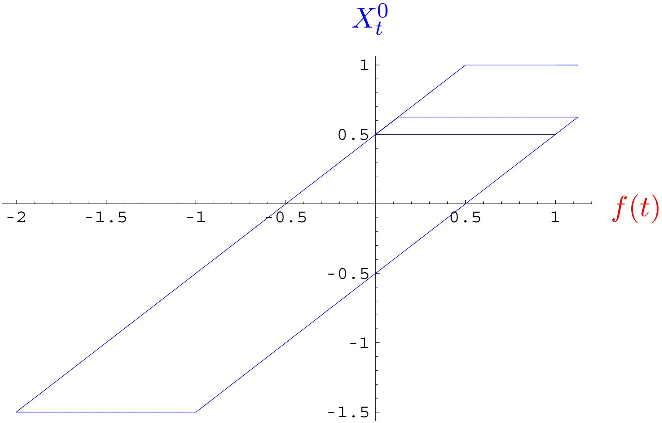
"Drainpipe rule" for solutions of (ODI)

X_t^0, $f(t)$, $\mathcal{A}^\gamma(f(t))$



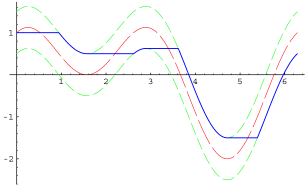
A typical deterministic, rate-independent $\mathbb{P}$-a.s. limit X^0, shown in blue. Loading $f(t) = \sin t + \cos 2t$ shown in red; "sticky attractor" $\mathcal{A}^\gamma(f(t))$ shown in green.

Hysteresis loops



Hysteresis loops for the deterministic, rate-independent $\mathbb{P}$-a.s. limit X^0, shown in blue. Again, $f(t) = \sin t + \cos 2t$.

Sticky attractor for X^0 dynamics



- The sticky attractor $\mathcal{A}^\gamma : \mathbb{R} \to 2^{\mathbb{R}}$:

$$\mathcal{A}^\gamma(F) := \left[\frac{F - \gamma^+}{\kappa}, \frac{F - \gamma^-}{\kappa} \right].$$

- *Attractor* in the sense that all trajectories lie in $\mathcal{A}^\gamma(f(t))$ for all $t > 0$, regardless of initial condition.
- *Sticky* in the sense that if a trajectory can remain stationary and stay inside $\mathcal{A}^\gamma(f(t))$, it will.

Strategy of proof of main theorem

- Identify the fixed-point set for the dynamics at scale $\varepsilon > 0$, some fixed landscape given by $\omega \in \Omega$, constant loading $f(t) \equiv 0$.
- Take a suitable limit of these sets as $\varepsilon \downarrow 0$, $\mathbb{P}$-almost surely losing ω-dependence along the way.
- No loading $\rightsquigarrow$ constant loading $\rightsquigarrow$ variable loading.
- Show that the limit "tube" $t \mapsto \mathcal{A}^\gamma(f(t))$ has the desired properties (sticky attractor) for the process X^0.

Limits of sets

Definition (Kuratowski (1966))

Let $(\mathbb{M}, d)$ be a metric space. Define the *Kuratowski limit inferior* of a family of subsets $\{A_\varepsilon \subseteq \mathbb{M}\}_{\varepsilon>0}$ to be

$$\mathrm{Li}_{\varepsilon\downarrow 0} A_\varepsilon := \left\{ x \in \mathbb{M} \,\middle|\, \limsup_{\varepsilon\downarrow 0} d_{\mathrm{H}}(x, A_\varepsilon) = 0 \right\},$$

where $d_{\mathrm{H}}(x, A_\varepsilon) := \inf_{y\in A_\varepsilon} d(x, y)$ is the usual Hausdorff semi-distance.

(The Kuratowski notions of limit superior (Ls) and limit (Lim) are not required in this analysis.)

Key lemma

$$\dot{X}_t^\varepsilon(\omega) = -\kappa X_t^\varepsilon(\omega) - g\left(\omega, \frac{X_t^\varepsilon(\omega)}{\varepsilon}\right) + f(\varepsilon t).$$

Lemma

Let $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ *be a doubly-reflected Wiener process and let*

$$A_\varepsilon^{g(\omega)}(0) := \left\{ x \in \mathbb{R} \,\middle|\, -\kappa x - g\left(\omega, \frac{x}{\varepsilon}\right) = 0 \right\},$$

the fixed-point set for the dynamics in the landscape $V(\cdot) + \varepsilon G(\omega, \cdot/\varepsilon)$ *at scale* $\varepsilon > 0$ *with no loading. Then*

$$\mathrm{Li}_{\varepsilon \downarrow 0} A_\varepsilon^{g(\omega)}(0) = \mathcal{A}^\gamma(0) \equiv \left[\frac{-\gamma^+}{\kappa}, \frac{-\gamma^-}{\kappa}\right] \text{ for } \mathbb{P}\text{-a.a. } \omega \in \Omega.$$

"The attractors for $\varepsilon > 0$ fill up the correct interval as $\varepsilon \downarrow 0$."

Sketch proof of key lemma

- Idea: intermediate value theorem + scaling.

Sketch proof of key lemma

- Idea: intermediate value theorem + scaling.
- Define "first return separations" $D_n(\omega)$ from γ^+ to γ^- and back to γ^+.
- Require (in both directions): sample-continuity of g, $D_n < +\infty$ $\mathbb{P}$-a.s., $\sum_n D_n = +\infty$ $\mathbb{P}$-a.s. and

$$\frac{D_n}{\sum_{i=0}^{n-1} D_i} \xrightarrow[n \to \infty]{} 0 \ \mathbb{P}\text{-a.s.} \qquad (\maltese)$$

Sketch proof of key lemma

- Idea: intermediate value theorem + scaling.
- Define "first return separations" $D_n(\omega)$ from γ^+ to γ^- and back to γ^+.
- Require (in both directions): sample-continuity of g, $D_n < +\infty$ $\mathbb{P}$-a.s., $\sum_n D_n = +\infty$ $\mathbb{P}$-a.s. and

$$\frac{D_n}{\sum_{i=0}^{n-1} D_i} \xrightarrow[n\to\infty]{} 0 \ \mathbb{P}\text{-a.s.} \qquad (✠)$$

- For $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ a doubly-reflected Wiener process, all the conditions (✠) are met (g sample-continuous with D_n IID, $\mathbb{E}[D_n] = 4|\gamma^+ - \gamma^-|^2$, $\mathrm{Var}[D_n] = 32|\gamma^+ - \gamma^-|^4$).

Sketch proof of key lemma

- Clearly, many more processes satisfy (✠), but a doubly-reflected Wiener process is a good prototype.
- In fact, something better is true: the conditions (✠) are *necessary and sufficient* to conclude that

$$\operatorname{Li}_{\varepsilon \downarrow 0} A_{\varepsilon}^{g(\omega)}(0) = \left[\frac{-\gamma^{+}}{\kappa}, \frac{-\gamma^{-}}{\kappa} \right] \ \mathbb{P}\text{-a.s.}$$

  - Argue from contradiction.
  - If any one of the conditions (✠) fails then there is a collection of "bad" landscapes of positive probability for which $\operatorname{Li}_{\varepsilon \downarrow 0} A_{\varepsilon}^{g(\omega)}(0)$ is not what we want.

Further lemmata

Lemma (Stickiness locally in time)

Let $0 \leq t_0 < t_1 < \infty$ and let I denote any interval from t_0 to t_1 with either end open or closed. $\mathbb{P}$-a.s., if $f|_I$ is bounded, and

$$X^0_{t_0} \in \mathcal{A}^\gamma(f(t)) \text{ for all } t \in I,$$

then $X^0_t = X^0_{t_0}$ for all $t \in I$.

Lemma (Right limit property)

Let $t_0 \geq 0$ be such that $f(t_0+)$ exists. Then

$$X^0_{t_0+} = X^0_{t_0} \diamond \mathcal{A}^\gamma(f(t_0+)) \ \mathbb{P}\text{-a.s.},$$

where $y \diamond A$ denotes the closest point of the interval $\bar{A}$ to y.

Main theorem revisited

Theorem (T.J.S.–F.T. (2006-07))

Let $f \in C^0([0, +\infty); \mathbb{R})$ *and let* $g : \Omega \times \mathbb{R} \to [\gamma^-, \gamma^+]$ *be any stochastic process. Then* g *satisfies (✠) if, and only if,* X^0 $\mathbb{P}$*-a.s. satisfies the deterministic ordinary differential inclusion*

$$-V'(X_t^0) + f(t) \in \partial \psi^\gamma(\dot{X}_t^0), \qquad \text{(ODI)}$$

where the dissipation $\psi^\gamma : \mathbb{R} \to [0, +\infty)$ *is given by*

$$\psi^\gamma(\dot{x}) := \begin{cases} \gamma^- \dot{x}; & \dot{x} \leq 0; \\ \gamma^+ \dot{x}; & \dot{x} \geq 0. \end{cases}$$

Some conclusions

- If one subscribes to the idea that rate-independent evolutions like (ODI) should arise as small-scale limits of deterministic evolutions in wiggly energies, our theorem shows that the precise choice of wiggle is not so important.
- "Homogenization without periodicity of the fast (microscale) process."

Further work

- Extension to more general spatial noise processes g?
- Extension to $\mathbb{R}^d$, $d \geq 1$? To infinite-dimensional spaces like $W^{k,p}(\mathcal{D}; \mathbb{R})$?

$$\dot{X}_t = -\nabla V(X_t) - \nabla G\left(\omega, \frac{X_t}{\varepsilon}\right) + f(\varepsilon t).$$

- Include the effects of a heat bath via a stochastic differential?

$$\dot{X}_t^\varepsilon(\omega_1, \omega_2) = -\nabla V(X_t^\varepsilon) - \nabla G\left(\omega_1, \frac{X_t^\varepsilon}{\varepsilon}\right) + \sigma(\varepsilon)\dot{W}_t(\omega_2).$$

Which "wins" as $\varepsilon \downarrow 0$? The diffusive or the stick-slip dynamics?