Optimal transport and dissipative systems

Deterministic limits for stochastic processes

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Introduction

Basic question

How does a dissipative system behave when placed in contact with a heat bath?

Aim

To use a time-incremental variational scheme to describe such systems as stochastic processes. Specifically, the variational scheme will arise as a perturbation/regularization of the deterministic variational set-up by an entropy-like term.

Main result

In the inertialess case, as time step $\downarrow 0$, we obtain deterministic gradient flows (with variational characterization).

Background

- R. Jordan, D. Kinderlehrer & F. Otto
  The variational formulation of the Fokker-Planck equation
  SIAM J. Math. Anal. 1998
- K.
  A phase-field model of dislocations in ductile single crystals
  Calif. Inst. of Technology 2003
- O. & R. Radovitzky 1999, O. & E.A. Repetto 1999, O. & L. Stainier 1999... and many more

Set-up: deterministic evolutions

- Consider a (first order) dissipative system in $\mathbb{R}^N$ as given by its equilibrium equations

$$\partial\Psi(\dot{x}) \ni -\operatorname{grad} E(t, x),$$

where
  - E is an energetic potential, possibly time-dependent if there is some external loading ℓ;
  - Ψ is a dissipation potential.
- What happens when this system is placed in contact with a heat bath?
- Strategy: perturb/regularize the discrete-time variational formulation of the equilibrium equations.

Set-up: incremental variational problem

- Time step $h > 0$; let $t_i := ih$.
- Work done moving along a "nice" path $\gamma \colon [t_i, t_{i+1}] \to \mathbb{R}^N$:

$$W(\gamma) := \underbrace{E(t_{i+1}, \gamma(t_{i+1})) - E(t_i, \gamma(t_i))}_{\text{energetic pot.}} + \underbrace{\int_{t_i}^{t_{i+1}} \Psi\big(\dot{\gamma}(t)\big)\,\mathrm{d}t}_{\text{dissipative pot.}}.$$

- Approximate the "work distance" $\inf_\gamma W(\gamma)$ by the *incremental cost/work function*

$$\begin{aligned} C\left(t_i, x_i, t_{i+1}, x_{i+1}\right) :=& E(t_{i+1}, x_{i+1}) - E(t_i, x_i) \\ &+ (t_{i+1} - t_i)\Psi\left(\frac{x_{i+1} - x_i}{t_{i+1} - t_i}\right). \end{aligned}$$

- Applying Euler-Lagrange to C gives the equilibrium equations.

Regularized optimal transport

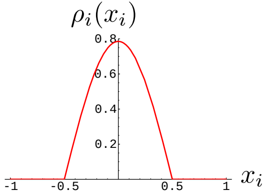
Heuristic definition

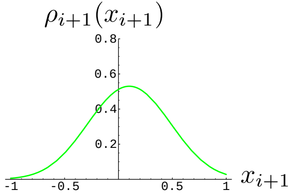
Given a time step $h > 0$, "temperature" parameter $\varepsilon > 0$ and cost function C, the *regularized optimal transport problem* is to find a joint probability density $u_{i+1}^{h,\varepsilon}(x_i, x_{i+1})$ on $(\mathbb{R}^N)^2$, with first marginal a given density $\rho_i^{h,\varepsilon}(x_i)$, minimizing

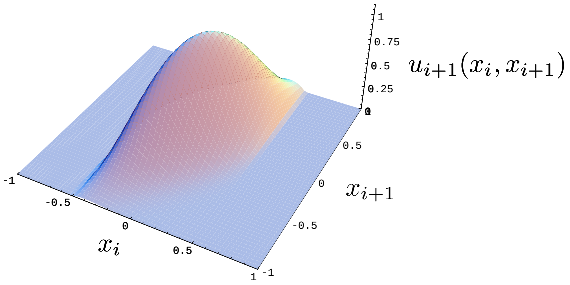
$$\iint_{\mathbb{R}^N} C(x_i, x_{i+1}) u_{i+1}^{h,\varepsilon}(x_i, x_{i+1}) + \varepsilon u_{i+1}^{h,\varepsilon}(x_i, x_{i+1}) \log u_{i+1}^{h,\varepsilon}(x_i, x_{i+1}) \, \mathrm{d}x_i \mathrm{d}x_{i+1}.$$

Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the *regularized optimal transport chain* $X^{h,\varepsilon} \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^N$ is the Markov chain whose transition probabilities solve a causal sequence of such single-step problems. (!)

Regularized optimal transport







Regularized optimal transport

Lemma

The single-step regularized optimal transport problem extends continuously from definition on $\mathcal{P}_{\mathrm{ac}}(\mathbb{R}^N)$ *to the space of all Radon measures on* $\mathbb{R}^N$.

Lemma (*cf.*, e.g. Jordan-Kinderlehrer '96)

A regularized optimal transport chain has transition probabilities $\mathbb{P}^{h,\varepsilon}_{i+1}(-|x_i)$ *with densities* $\rho^{h,\varepsilon}_{i+1}(-|x_i)$ *with respect to Lebesgue measure given by*

$$\rho^{h,\varepsilon}_{i+1}(x_{i+1}|x_i) = \frac{\exp\left(-\frac{1}{\varepsilon}C(x_i, x_{i+1})\right)}{\int_{\mathbb{R}^N} \exp\left(-\frac{1}{\varepsilon}C(x_i, x_{i+1})\right)\, \mathrm{d}x_{i+1}}.$$

Strategy/programme

Definition

Let $\bar{X}^{h,\varepsilon} \colon \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ denote the continuous, piecewise affine interpolation

$$\bar{X}_t^{h,\varepsilon} := X_{\lfloor t/h \rfloor}^{h,\varepsilon} + \left(\frac{t}{h} - \left\lfloor \frac{t}{h} \right\rfloor \right) \left(X_{\lfloor t/h \rfloor + 1}^{h,\varepsilon} - X_{\lfloor t/h \rfloor}^{h,\varepsilon} \right).$$

Thus, $\bar{X}^{h,\varepsilon}$ is a measurable function (random variable)

$$\bar{X}^{h,\varepsilon} \colon \Omega \to C^0\big(\mathbb{R}_{\geq 0}; \mathbb{R}^N\big),$$

and we can consider the law (push-forward measure) $\big(\bar{X}^{h,\varepsilon}\big)_*(\mathbb{P})$ on the space of continuous paths in $\mathbb{R}^N$, take (weak) limits as $h \downarrow 0$ & c.

Strategy/programme

- Get as much information as we can about the transition densities $\rho_{i+1}^{h,\varepsilon}(-|x_i)$.
- Hope that the increments follow some "decent" distribution

$$\Delta X_{i+1}^{h,\varepsilon} \equiv X_{i+1}^{h,\varepsilon} - X_i^{h,\varepsilon} \sim \mathrm{Distrib}\big(X_i^{h,\varepsilon}, h, \varepsilon, E, \Psi\big).$$

- Apply a central limit theorem to say something about the distribution of the continuous-time increments as $h \downarrow 0$:

$$\bar{X}_t^{h,\varepsilon} - \bar{X}_s^{h,\varepsilon} \approx X_{\lfloor t/h \rfloor}^{h,\varepsilon} - X_{\lfloor s/h \rfloor}^{h,\varepsilon} = \sum_{i=\lfloor s/h \rfloor}^{\lfloor t/h \rfloor - 1} \Delta X_{i+1}^{h,\varepsilon} \xrightarrow[h\downarrow 0]{} \mathrm{Normal}(?, ?).$$

- Conclude something interesting about the limiting measure on path space.

An illustrative example: Itō diffusions

Relationship with noise

Why should this optimal transport scheme be considered a valid model for $\partial\Psi(\dot{x}) \ni -\operatorname{grad} E(t, x) +$ "noise"?

"Proof by example" (linear kinetics and Itō diffusions)

On $\mathbb{R}^N$, consider:

- a coercive, C^2 energetic potential $E\colon \mathbb{R}^N \to \mathbb{R}$;
- a viscous (i.e. 2-homogeneous), isotropic dissipative potential $\Psi\colon \mathbb{R}^N \to \mathbb{R}$, $\Psi(\dot{x}) := \frac{1}{2}|\dot{x}|^2$.

The key observation about the increments is that

$$X_{i+1}^{h,\varepsilon} - X_i^{h,\varepsilon} \sim \mathrm{Normal}\left(-h \operatorname{grad} E\big(X_i^{h,\varepsilon}\big), \varepsilon h\right).$$

An illustrative example: Itō diffusions

Theorem (*cf.* Jordan-Kinderlehrer-Otto '98)

Given $\varepsilon > 0$*, let* $X^{h,\varepsilon}$ *be the optimal transport chain for the potentials on the previous slide starting at* $x_0 \in \mathbb{R}^N$*. Then*

$$\bar{X}^{h,\varepsilon} \xrightharpoonup[h \downarrow 0]{} Y^{\varepsilon},$$

where

$$\begin{cases} \dot{Y}_t^{\varepsilon} = -\operatorname{grad} E(Y_t^{\varepsilon}) + \sqrt{\varepsilon}\dot{W}_t; \\ Y_0^{\varepsilon} = x_0. \end{cases}$$

I.e. $\mathbb{P}\big[\bar{X}^{h,\varepsilon} \in B\big] \xrightarrow[h \downarrow 0]{} \mathbb{P}[Y^{\varepsilon} \in B]$ *for all Borel* $B \subseteq C^0\big(\mathbb{R}_{\geq 0}; \mathbb{R}^N\big)$*.*

Dry friction in a flat energetic landscape

Consider the following toy model for an inertialess slider subject to time-dependent loading ℓ, dissipation Ψ and a heat bath with "temperature" ε:

- time-dependent energetic potential $E(t, x) = -\ell(t) \cdot x$;
  - Lipschitz forcing/loading $\ell \colon [0, T] \to \mathbb{R}^N$;
- 1-homogeneous, positive-definite dissipative potential $\Psi \colon \mathbb{R}^N \to [0, +\infty)$;
  - e.g. isotropic case $\Psi(\dot{x}) = \sigma|\dot{x}|$;
- More generally, call $\sigma := \min_{|\dot{x}|=1} \Psi(\dot{x})$ the *critical load*. Consider loadings ℓ with $\|\ell\|_{C^0} < \sigma$.

Mean and variance

- Consider the increment $\Delta X_{i+1}^{h,\varepsilon} \equiv X_{i+1}^{h,\varepsilon} - X_i^{h,\varepsilon}$.
- The mean scales like ε, but the variance scales like ε^2, and both are independent of h:

$$\mathbb{E}\left[\Delta X_{i+1}^{h,\varepsilon} \middle| X_i^{h,\varepsilon}\right] = \mathbb{E}\left[\Delta X_{i+1}^{h,\varepsilon}\right] = \varepsilon \xi_{\mathrm{eff}}(t_{i+1});$$

$$\mathrm{Var}\left[\Delta X_{i+1}^{h,\varepsilon} \middle| X_i^{h,\varepsilon}\right] = \mathrm{Var}\left[\Delta X_{i+1}^{h,\varepsilon}\right] \leq \mathrm{const} \cdot \varepsilon^2;$$

where the *effective load/drift* is

$$\xi_{\mathrm{eff}}(t_{i+1}) = \frac{\int_{\mathbb{R}^N} z \exp\big(z \cdot \ell(t_{i+1}) - \Psi(z)\big)\, \mathrm{d}z}{\int_{\mathbb{R}^N} \exp\big(z \cdot \ell(t_{i+1}) - \Psi(z)\big)\, \mathrm{d}z}$$

- Ignoring the variance part, this looks rather like an Euler approximation for the ODE $\dot{Y}_t = \xi_{\mathrm{eff}}(t)$ with time step ε.

Mean and variance

Definition

An *effective kinetic potential* for ξ_{eff} is given by

$$\varphi_{\text{eff}}(t, x) = -\xi_{\text{eff}}(t) \cdot x.$$

In the isotropic case $\Psi(\dot{x}) = \sigma|\dot{x}|$ (K. '03),

- ΔX_{i+1} is a Laplace random variable;
- the effective load is

$$\xi_{\text{eff}}(t) = \frac{(N+1)\ell(t)}{\sigma^2 - |\ell(t)|^2};$$

- the effective kinetic potential is

$$\varphi_{\text{eff}}(t, x) = -\frac{(N+1)\ell(t) \cdot x}{\sigma^2 - |\ell(t)|^2}.$$

Why take $\varepsilon \propto$ time step?

- It sits well with comparison with the implicit Euler scheme.
- Dimensionally, this corresponds to energy input at power ε / h.
- Alternatively, if one were to naïvely apply the central limit theorem:

$$\bar{X}_t^{h,\varepsilon} - x_0 \approx X_{\lfloor t/h \rfloor}^{h,\varepsilon} - x_0 = \sum_{i=0}^{\lfloor t/h \rfloor - 1} \Delta X_{i+1}^{h,\varepsilon}$$

$$\sim \mathrm{Normal}\left(\sum_{i=0}^{\lfloor t/h \rfloor - 1} \mathbb{E}\big[\Delta X_{i+1}^{h,\varepsilon}\big], \sum_{i=0}^{\lfloor t/h \rfloor - 1} \mathrm{Var}\big[\Delta X_{i+1}^{h,\varepsilon}\big] \right)$$

but, as $h \downarrow 0$, a sum of $\lfloor t/h \rfloor$ terms of order ε

- diverges if ε is of order h^p, $p < 1$;
- converges to 0 if ε is of order h^p, $p > 1$.

Theorem: deterministic limit for flat potential

Theorem (K.-O.-S.-T. '07)

For $\ell \in C^{0,1}\big([0,T];\mathbb{R}^N\big)$ *with* $\|\ell\|_{C^0} < \sigma$ *and* $\theta > 0$*, let* $Y^\theta \colon [0,T] \to \mathbb{R}^N$ *solve the deterministic* IVP

$$\begin{cases} \dot{Y}^\theta_t = \theta \xi_{\mathrm{eff}}(t) \equiv -\theta \operatorname{grad} \varphi_{\mathrm{eff}}\big(t, Y^\theta_t\big); \\ Y^\theta_0 = x_0. \end{cases}$$

Then, for $\lambda > 0$*, taking* $\varepsilon = \theta h$*,*

$$\mathbb{P}\left[\big\| \bar{X}^{h,\theta h} - Y^\theta \big\|^2_{C^0} \geq \lambda \right] \leq \frac{\mathrm{const} \cdot \theta^2 T h}{\lambda^2},$$

I.e., $\bar{X}^{h,\theta h} \to Y^\theta$ *in probability (and hence in law) in path space* $C^0\big([0,T];\mathbb{R}^N\big)$ *(with the supremum norm).*

Corollary: minimum principle

Corollary (*via* Brézis-Ekeland principle)

For $\ell \in C^{0,1}\big([0,T];\mathbb{R}^N\big)$ *with* $\|\ell\|_{C^0} < \sigma$ *and* $\theta > 0$,

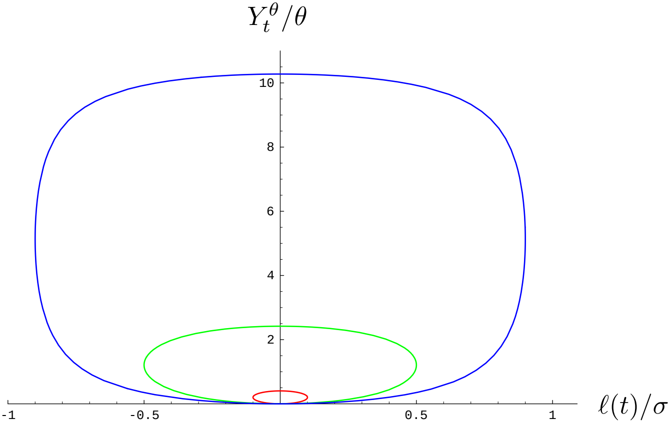
$$\bar{X}^{h,\theta h} \xrightarrow[h\downarrow 0]{\mathbb{P}} Y^\theta \text{ in } C^0\big([0,T];\mathbb{R}^N\big),$$

where Y^θ *is the unique minimizer of the action functional* $\mathscr{S}$ *given by*

$$\mathscr{S}[Y] := \int_0^T \mathscr{L}\big(t, Y_t, \dot{Y}_t\big)\,\mathrm{d}t,$$

$$\mathscr{L}\big(t,x,v\big) := \frac{1}{2}\big|v - \theta\xi_{\text{eff}}(t)\big|^2.$$

The deterministic limit for flat potential



Response Y_t^θ to loading $\ell(t) = \ell_0 \sin t$ for
$\ell_0 = 0.1\sigma$ (red), 0.5σ (green), 0.9σ (blue),
isotropic dissipation $\Psi(\dot{x}) = \sigma|\dot{x}|$.

Andrade's $t^{1/3}$ creep law

- $Y_t = \frac{\text{length}}{\text{ref. length}}$ for a material sample subjected to constant load $\ell \ll$ yield stress of the material.
- Linear strain hardening:

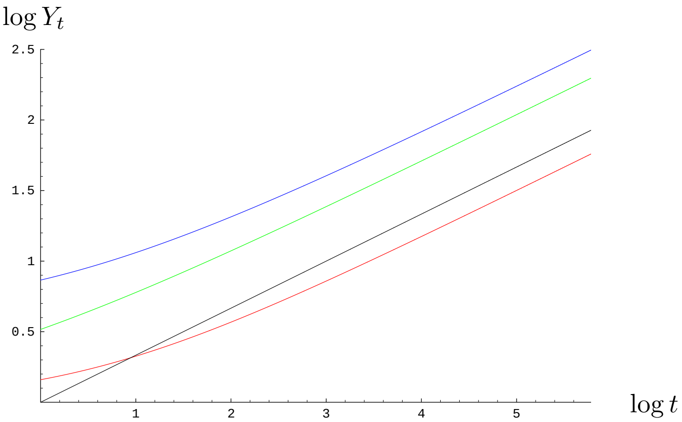
$$\sigma = \sigma_0 Y.$$

- For ℓ small in comparison to σ,

$$\dot{Y}_t = \frac{2\theta\ell}{\sigma_0^2 Y_t^2 - \ell^2} \approx \frac{2\theta\ell}{\sigma_0^2 Y_t^2}.$$

- $\rightsquigarrow$ Andrade's creep law: strain grows like $t^{1/3}$.
- Hardening exponent $\beta \rightsquigarrow$ creep exponent $1 + 2\beta$.

Andrade's $t^{1/3}$ creep law



Log-log plot of strain Y_t (colour) against time t.
$Y_0 = 1$; $\theta = 1$; $\sigma_0 = 1$; $\ell = 0.1$ (red), 0.5 (green), 0.9 (blue).
$t \mapsto t^{1/3}$ in black.

Dry friction in a parabolic energetic landscape

Consider the the following toy model for an inertialess gradient flow in E subject to dissipation Ψ and a heat bath with "temperature" ε

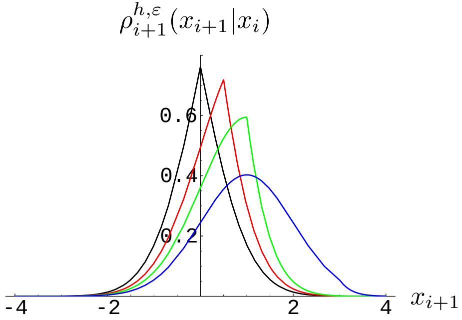
- energetic potential: $E(t, x) := \frac{1}{2} x \cdot Ax - \ell(t) \cdot x$ with $A = \mathrm{Hess}(E)$ symmetric and positive-definite;
- dissipative potential: $\Psi \colon \mathbb{R}^N \to \mathbb{R}$ 1-homogeneous with critical load $\sigma = \min_{|\dot{x}|=1} \Psi(\dot{x}) > 0$;
- "interesting" / *admissible region* $\mathcal{B}$,

$$\begin{aligned} \mathcal{B}(\ell(t)) &= \left\{ x \in \mathbb{R}^N \middle| |\ell(t) - Ax| < \sigma \right\} \\ &= A^{-1}\ell(t) + A^{-1}\mathbb{B}_\sigma(0). \end{aligned}$$

I.e., the previous model with the slider restrained by a spring with elasticity matrix A.

Mode

For each $x_i \in \mathbb{R}^N$, the transition density $\rho^{h,\varepsilon}_{i+1}(-|x_i)$ is unimodal: the mode is precisely the (unique) closest point of $\overline{\mathcal{B}(\ell(t_{i+1})}$ to x_i.



$\rho^{h,\varepsilon}_{i+1}(x_{i+1}|x_i)$ for $x_i = 0$ (black), $\frac{1}{2}$ (red), 1 (green), 3 (blue), $\varepsilon = 1$, $\sigma = 1$, $\ell = 0$, $\mathcal{B} = (-1, 1) \subsetneq \mathbb{R}$.

Mean and variance

- Same ε-ε^2 scaling for mean and variance of $\Delta X_{i+1}^{h,\varepsilon}$, but the effective load is no longer independent of ε or prior position. For $X_i^{h,\varepsilon} \in \mathcal{B}(\ell(t_{i+1}))$,

$$\mathbb{E}\left[\Delta X_{i+1}^{h,\varepsilon} \middle| X_i^{h,\varepsilon}\right] = \varepsilon \xi^{\varepsilon}\left(t_{i+1}, X_i^{h,\varepsilon}\right);$$

$$\operatorname{Var}\left[\Delta X_{i+1}^{h,\varepsilon} \middle| X_i^{h,\varepsilon}\right] \leq \frac{\text{const}}{\left(\sigma^2 - \left|\ell(t_{i+1}) - AX_i^{h,\varepsilon}\right|^2\right)^2} \varepsilon^2 + o(\varepsilon^2);$$

where

$$\xi^{\varepsilon}(t,x) := \frac{\displaystyle\int_{\mathbb{R}^N} z \exp\left(-\left(z \cdot (Ax - \ell(t)) + \Psi(z) + \tfrac{\varepsilon}{2} z \cdot Az\right)\right) \, \mathrm{d}z}{\displaystyle\int_{\mathbb{R}^N} \exp\left(-\left(z \cdot (Ax - \ell(t)) + \Psi(z) + \tfrac{\varepsilon}{2} z \cdot Az\right)\right) \, \mathrm{d}z}.$$

- Given $X_i \notin \bar{\mathcal{B}}$, $X_{i+1}^{h,\varepsilon}$ has mean the nearest point of $\partial\mathcal{B}$ and variance of order ε.

Effective gradient structure

Lemma

For $\varepsilon \geq 0$, A symmetric and positive-definite, $\xi^\varepsilon(t, x)$ is the gradient in x of

$$\varphi^\varepsilon(t, x) := \log \int_{\mathbb{R}^N} \exp\left(-\left(z \cdot (Ax - \ell(t)) + \Psi(z) + \tfrac{\varepsilon}{2} z \cdot Az\right)\right) \, \mathrm{d}z,$$

with respect to the the constant Riemannian metric A, i.e.

$$\xi^\varepsilon(t, x) = -\operatorname{grad}_A \varphi^\varepsilon(t, x) \equiv -A^{-1} \mathrm{d}\varphi^\varepsilon(t, x).$$

Definition

As before, call $\xi_{\mathrm{eff}} := \xi^0$ the *effective load* and $\varphi_{\mathrm{eff}} := \varphi^0$ the *effective kinetic potential.*

Effective gradient structure

In the isotropic case $\Psi(\dot{x}) = \sigma|\dot{x}|$,

- the effective load is

$$\xi_{\text{eff}}(t,x) = \frac{(N+1)(\ell(t) - Ax)}{\sigma^2 - |\ell(t) - Ax|^2};$$

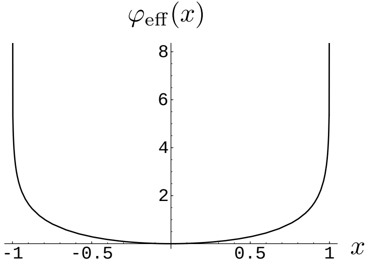
- the effective kinetic potential is

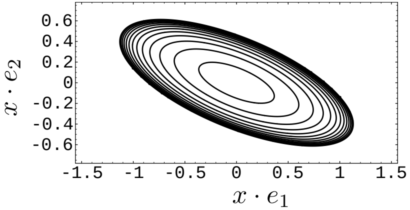
$$\varphi_{\text{eff}}(t,x) = -\frac{N+1}{2} \log\left(\sigma^2 - |\ell(t) - Ax|^2\right);$$

- and

$$\xi_{\text{eff}}(t,x) = -\operatorname{grad}_A \varphi_{\text{eff}}(t,x) \equiv -A^{-1}\mathrm{d}\varphi_{\text{eff}}(t,x).$$

Effective gradient structure





The effective potential φ_{eff} in dimensions $N = 1, 2$ for $\ell(t) = 0$, $\Psi(\dot{x}) = \sigma|\dot{x}|$. Note the blow-up at the boundary of the "admissible region", i.e. where net force equals dissipation.

Theorem: deterministic limit for positive-definite parabolic potential

Theorem (K.-O.-S.-T. '07)

Let $\theta > 0$; *let* $Y^\theta \colon [0, T] \to \mathbb{R}^N$ *solve the deterministic* IVP

$$\begin{cases} \dot{Y}_t^\theta = \theta \xi_{\text{eff}}\big(t, Y_t^\theta\big) \equiv -\theta \,\mathrm{grad}_A \,\varphi_{\text{eff}}\big(t, Y_t^\theta\big); \\ Y_0^\theta = x_0 \in \mathcal{B}(\ell(0)); \end{cases}$$

Then, for $\lambda > 0$, *taking* $\varepsilon = \theta h$,

$$\mathbb{P}\left[\big\| \bar{X}^{h, \theta h} - Y^\theta \big\|_{C^0}^2 \geq \lambda \right] \in O(h) \text{ as } h \downarrow 0.$$

I.e., $\bar{X}^{h,\theta h} \to Y^\theta$ *in probability (and hence in law) in path space* $C^0\big([0, T]; \mathbb{R}^N\big)$ *(with the supremum norm).*

Corollary: minimum principle

Corollary

With the same assumptions as before,

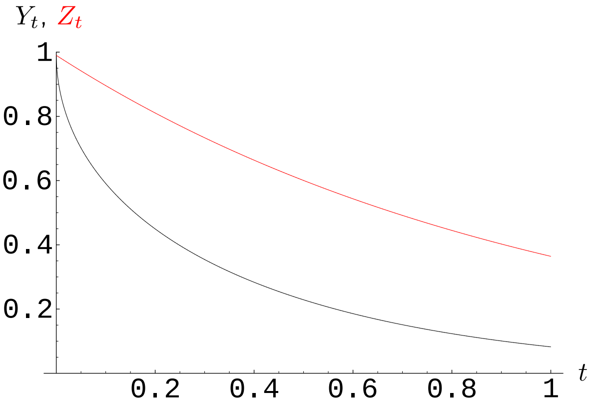
$$\bar{X}^{h,\theta h} \xrightarrow[h \downarrow 0]{\mathbb{P}} Y^{\theta} \text{ in } C^0([0,T];\mathbb{R}^N),$$

where Y^{θ} is the unique minimizer of the action functional $\mathscr{S}$ given by

$$\mathscr{S}[Y] := \int_0^T \mathscr{L}(t, Y_t, \dot{Y}_t)\, \mathrm{d}t,$$

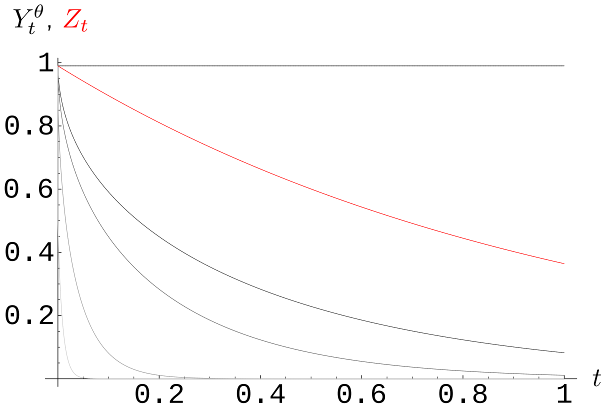
$$\mathscr{L}(t,x,v) := \frac{1}{2}\left|v - \theta\xi_{\mathrm{eff}}(t,x)\right|^2.$$

The deterministic gradient flow, $N = 1$



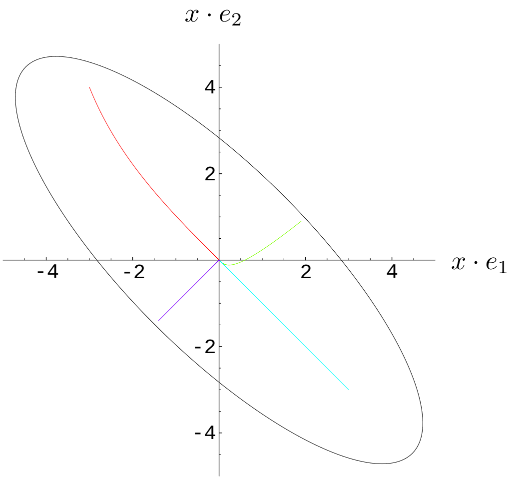
Comparison of the limiting gradient flow $\dot{Y}_t = -\varphi'_{\mathrm{eff}}(Y_t)$ (black) with the "frictionless" gradient flow $\dot{Z}_t = -AZ_t$ (red).

The deterministic gradient flow, $N = 1$



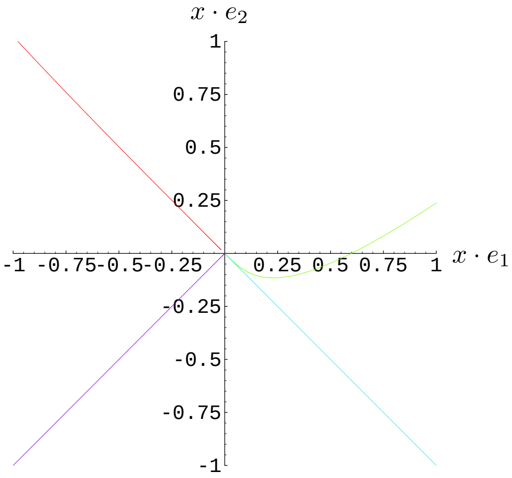
Y_t^θ in greyscale. From darkest to lightest: $\theta = 0, 1, 2, 10, 50$.
$\theta = 0 \leftrightarrow$ rate-independent limit at zero temperature.

The deterministic gradient flow, $N = 2$



In black, the "yield surface" $\partial\mathcal{B} = \sigma A^{-1}(\mathbb{S}^1)$. In colour, trajectories for Y_t, $t \in [0, 100]$, for various initial data. Note the approach along the eigenvector of A^{-1} with largest eigenvalue.

The deterministic gradient flow, $N = 2$



In black, the "yield surface" $\partial\mathcal{B} = \sigma A^{-1}(\mathbb{S}^1)$. In colour, trajectories for Y_t, $t \in [0, 100]$, for various initial data. Note the approach along the eigenvector of A^{-1} with largest eigenvalue.

Conclusions

- The optimal transport scheme can describe stochastic processes such as Itō diffusions.
- It can also describe dry-friction-like systems, at least when inertia is neglected.
- We extract a deterministic limit (with gradient structure and minimum principle) from an *a priori* stochastic system by exploiting the dependence of the regularization parameter on the length of then time step.
- The resulting evolutions encompass physical processes such as creep.

Directions for future/ongoing research

- How is $\theta = \frac{\varepsilon}{h}$ related to physical temperature?
- Energies with positive-*semi*-definite and/or non-constant Hessian A?
- Incorporation of inertial effects?
  - Heated harmonic oscillators? Belt friction?
  - Other slipping processes, e.g. earthquakes, fluid invasion & *c.*?
- Can the analysis be extended to infinite-dimensional and/or curved state spaces and make better use of the general set-up for optimal transport chains?

Main point

Despite connections with steepest descent methods, the optimal transport framework is somewhat *ad hoc*: can it be justified e.g. from wiggly energies with stochastic noise?

Directions for future/ongoing research

Theorem (Abeyaratne-Chu-James '96; S.-T. '07)

For a "decent" potential $V : \mathbb{R} \to \mathbb{R}$, let X^ε satisfy the forced wiggly gradient flow

$$\dot{X}^\varepsilon(t) = -V'\big(X^\varepsilon(t)\big) - g\left(\frac{X^\varepsilon(t)}{\varepsilon}\right) + \ell(\varepsilon t).$$

Then $X^0(t) := \lim_{\varepsilon \downarrow 0} X^\varepsilon(t/\varepsilon)$ satisfies the inclusion

$$\partial\big|\dot{X}^0(t)\big| \ni -V'\big(X^0(t)\big) + \ell(t)$$

iff $g \colon \mathbb{R} \to [-1, 1]$ attains its bounds "often enough" (e.g. periodic [A.-C.-J.], or almost any sample path of doubly reflected Brownian motion [S.-T.]).

Directions for future/ongoing research

Conjecture/Hope

For "decent" potentials V, perturbations g and the right scaling $\sigma = \sigma(\varepsilon)$, solutions of

$$\dot{X}^{\varepsilon}(t) = -V'\big(X^{\varepsilon}(t)\big) - g\left(\frac{X^{\varepsilon}(t)}{\varepsilon}\right) + \ell(t) + \sigma(\varepsilon)\dot{W}(t)$$

converge as $\varepsilon \downarrow 0$ to the trajectory predicted by the optimal transport chain method — noise "activates" the system in a deterministic way in the admissible (sticking) region.